Frequency Estimation in Single-Frequency Complex Tone Problem from Limited Number of Noisy Observations.

Arezoo Rajabi, Shashini De Silva

Abstract

In this report, we discuss frequency estimation in single-frequency complex tone problem from limited number of noisy observations. To estimate this parameter, we used two different estimators and compare them with each other. Also, we derived the Carmer-Rao lower bounds for all parameters of multiple-frequency complex tone problem. Also, we evaluate performance of each estimator in different noise level. As shown, maximum likelihood estimator works well for noise with small variance. For noise with large variance, the empirical mean square error of both estimators are high.

1. Introduction

The problem of estimating the parameters of single-frequency tone has lot of application such as data set testing (Steven (1993)). This project deals with frequency estimation in single-frequency complex tone problem. In general, the signal with multiple-frequency has the following form:

$$y(n) = \sum_{k=1}^{K} c_k e^{j2\pi f_k (n-\frac{N+1}{2})} + v(n) = \sum_{k=1}^{K} (a_k + jb_k) e^{j2\pi f_k (n-\frac{N+1}{2})} + v(n), \quad n = 1, \cdots, N \quad (1)$$

where $c_k = a_k + jb_k$ is complex amplitude coefficient, a_k and b_k are the real-valued, real and imaginary components of the complex coefficient c_k. And, f_k is the frequency of the k^{th} component and (n) for $n = 1, 2, ..., N$ are i.i.d. complex Gaussian random variables ($v(n) \sim \mathcal{CN}\ (0, \sigma^2)$) representing the additive noise. Here, we assumed that all parameters of c_k, f_k are unknown. We define the parameter vector θ and observation vector Y as below:

$$\begin{aligned} \theta &= \begin{bmatrix} a_1 & \cdots & a_K & b_1 \cdots & b_K & f_1 & \cdots & f_K \end{bmatrix}^T \\ Y &= \begin{bmatrix} y(1) & y(2) & \cdots & y(n) \end{bmatrix}^T \end{aligned} \quad (2)$$

The noises $v(n)$ are i.i.d and come from $\mathcal{CN}\ (0, \sigma^2)$. Therefore,
$Y \sim \mathcal{CN}\ (\mu, \sigma^2 I_{N\times N})$ where μ define as follow:

$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_N \end{bmatrix} \qquad \mu_n = \sum_{k=1}^{K} (a_k + jb_k) e^{j2\pi f_k (n-\frac{N+1}{2})} \quad (3)$$

Therefore, an explicit form for PDF of y given θ ($f_Y(y|\theta)$) can be written as below:

$$f(y(n)|\theta) = \frac{1}{\pi\sigma^2} \exp\left[\frac{-1}{\sigma^2} |y(n) - \mu_n|^2\right] \quad (4)$$

Accordingly, $f_Y(Y|\theta)$ is obtained as below:

$$f_Y(y|\theta) = \sqcap_{n=1}^{N} f_Y(y(n), \theta) = \frac{1}{\pi^N \sigma^{2N}} \exp\left[-\sum_{n=1}^{N} \frac{1}{\sigma^2} |y(n) - \mu_n|^2\right] \quad (5)$$

2. Cramer-Rao Lower Bound

In this section, we tend to compute the FIM and the CRLB matrix. To simplify the derivation, use the well-known result of the CRLB for a Complex Gaussian process (Steven (1993)).

$$FIM_{ij} = tr[C_y^{-1}(\theta)\frac{\partial C_y(\theta)}{\partial \theta_i}C_y^{-1}(\theta)\frac{\partial C_y(\theta)}{\partial \theta_j}] + 2Re[\frac{\partial \mu^H(\theta)}{\theta_i}C_y^{-1}(\theta)\frac{\partial \mu^H(\theta)}{\theta_j}] \tag{6}$$

Because, the measurements $(y(1), \cdots, y(N))$ are independent and the variance of each measurements is known, therefore C_y is a diagonal matrix whose entries are not function of θ.

$$C_y = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}_{N \times N} \tag{7}$$

Therefore FIM for our problem gets simplified as given below

$$FIM_{lm} = 2Re[\frac{\partial \mu^H(\theta)}{\theta_l}C_y^{-1}(\theta)\frac{\partial \mu(\theta)}{\theta_m}] \tag{8}$$

We have three different type of parameters a_k, b_k and f_k. Therefore, we need to compute 7 different types of FIM.

2.1 FIM for two $a_k's(a_l, a_m)$

$$FIM_{lm} = 2Re[\frac{\partial \mu^H}{a_l}C_y^{-1}\frac{\partial \mu}{a_m}] = 2Re(\begin{bmatrix} e^{j2\pi f_l(1-\frac{N+1}{2})} \\ e^{j2\pi f_l(2-\frac{N+1}{2})} \\ \vdots \\ e^{j2\pi f_l(N-\frac{N+1}{2})} \end{bmatrix}^H \sigma^{-2} \begin{bmatrix} e^{j2\pi f_m(1-\frac{N+1}{2})} \\ e^{j2\pi f_m(2-\frac{N+1}{2})} \\ \vdots \\ e^{j2\pi f_m(N-\frac{N+1}{2})} \end{bmatrix})$$

$$= 2Re(\begin{bmatrix} e^{-j2\pi f_l(1-\frac{N+1}{2})} \\ e^{-j2\pi f_l(2-\frac{N+1}{2})} \\ \vdots \\ e^{-j2\pi f_l(N-\frac{N+1}{2})} \end{bmatrix}^T \sigma^{-2} \begin{bmatrix} e^{j2\pi f_m(1-\frac{N+1}{2})} \\ e^{j2\pi f_m(2-\frac{N+1}{2})} \\ \vdots \\ e^{j2\pi f_m(N-\frac{N+1}{2})} \end{bmatrix}) = \frac{2}{\sigma^2}Re(\sum_{n=1}^{N} e^{j2\pi(n-\frac{N+1}{2})(f_m-f_l)}) =$$

$$\frac{2}{\sigma^2}Re(\sum_{n=0}^{N-1} e^{j2\pi(n-\frac{N+3}{2})(\Delta f)}) = \frac{2}{\sigma^2}Re(e^{-j\pi(N+3)\Delta f}\sum_{n=0}^{N-1} e^{j2\pi n\Delta f}) =$$

$$\frac{2}{\sigma^2}Re(e^{-j\pi(N+3)\Delta f}\frac{\sin(N\pi\Delta f)}{\sin(\pi\Delta f)}e^{j2\pi\Delta f\frac{(N-1)}{2}}) = \frac{2}{\sigma^2}\cos(4\pi\Delta f)\frac{\sin(N\pi\Delta f)}{\sin(\pi\Delta f)}$$

$$FIM_{lm} = \begin{cases} \frac{2N}{\sigma^2}, & \text{if } m = 1, \\ \frac{2}{\sigma^2}\frac{\sin(N\pi\Delta f)}{\sin(\pi\Delta f)}\cos(4\pi\Delta f), & \text{otherwise.} \end{cases} \tag{9}$$

2.2 FIM for two $b_k's$ (b_l, b_m)

$$FIM_{lm} = 2Re[\frac{\partial \mu^H}{b_l} C_y^{-1} \frac{\partial \mu}{b_m}] = 2Re(\begin{bmatrix} je^{j2\pi f_l(1-\frac{N+1}{2})} \\ je^{j2\pi f_l(2-\frac{N+1}{2})} \\ \vdots \\ je^{j2\pi f_l(N-\frac{N+1}{2})} \end{bmatrix}^H \sigma^{-2} \begin{bmatrix} je^{j2\pi f_m(1-\frac{N+1}{2})} \\ je^{j2\pi f_m(2-\frac{N+1}{2})} \\ \vdots \\ je^{j2\pi f_m(N-\frac{N+1}{2})} \end{bmatrix})$$

$$= 2Re(\begin{bmatrix} -je^{-j2\pi f_l(1-\frac{N+1}{2})} \\ -je^{-j2\pi f_l(2-\frac{N+1}{2})} \\ \vdots \\ -je^{-j2\pi f_l(N-\frac{N+1}{2})} \end{bmatrix}^T \sigma^{-2} \begin{bmatrix} je^{j2\pi f_m(1-\frac{N+1}{2})} \\ je^{j2\pi f_m(2-\frac{N+1}{2})} \\ \vdots \\ je^{j2\pi f_m(N-\frac{N+1}{2})} \end{bmatrix}) = \frac{2}{\sigma^2} Re(\sum_{n=1}^{N} e^{j2\pi(n-\frac{N+1}{2})(f_m-f_l)}) =$$

$$\frac{2}{\sigma^2} Re(e^{-j\pi(N+3)\Delta f} \sum_{n=0}^{N-1} e^{j2\pi n\Delta f}) = \frac{2}{\sigma^2} Re(e^{-j\pi(N+3)\Delta f} \frac{\sin(N\pi\Delta f)}{\sin(\pi\Delta f)} e^{i2\pi\Delta f\frac{(N-1)}{2}}) =$$

$$\frac{2}{\sigma^2} \cos(4\pi\Delta f) \frac{\sin(N\pi\Delta f)}{\sin(\pi\Delta f)}$$

$$FIM_{lm} = \begin{cases} \frac{2N}{\sigma^2}, & \text{if } m = 1, \\ \frac{2}{\sigma^2} \frac{\sin(N\pi\Delta f)}{\sin(\pi\Delta f)} cos(4\pi\Delta f), & \text{otherwise.} \end{cases} \tag{10}$$

2.3 FIM for two $f_k's$ (f_l, f_m)

$$FIM_{lm} = 2Re[\frac{\partial \mu^H}{b_l} C_y^{-1} \frac{\partial \mu}{b_m}] =$$

$$2Re(\begin{bmatrix} j2\pi(1-\frac{N+1}{2})(a_l+jb_l)e^{j2\pi f_l(1-\frac{N+1}{2})} \\ j2\pi(2-\frac{N+1}{2})(a_l+jb_l)e^{j2\pi f_l(2-\frac{N+1}{2})} \\ \vdots \\ j2\pi(N-\frac{N+1}{2})(a_l+jb_l)e^{j2\pi f_l(2-\frac{N+1}{2})} \end{bmatrix}^H \sigma^{-2} \begin{bmatrix} j2\pi(1-\frac{N+1}{2})(a_m+jb_m)e^{j2\pi f_m(1-\frac{N+1}{2})} \\ j2\pi(2-\frac{N+1}{2})(a_m+jb_m)e^{j2\pi f_m(2-\frac{N+1}{2})} \\ \vdots \\ j2\pi(N-\frac{N+1}{2})(a_m+jb_m)e^{j2\pi f_m(N-\frac{N+1}{2})} \end{bmatrix})$$

$$= \frac{2}{\sigma^2} Re(\begin{bmatrix} -j2\pi(1-\frac{N+1}{2})(a_l-jb_l)e^{-j2\pi f_l(1-\frac{N+1}{2})} \\ -j2\pi(2-\frac{N+1}{2})(a_l-jb_l)e^{-j2\pi f_l(2-\frac{N+1}{2})} \\ \vdots \\ -j2\pi(N-\frac{N+1}{2})(a_l-jb_l)e^{-j2\pi f_l(N-\frac{N+1}{2})} \end{bmatrix}^T \begin{bmatrix} j2\pi(1-\frac{N+1}{2})(a_m+jb_m)e^{j2\pi f_m(1-\frac{N+1}{2})} \\ j2\pi(2-\frac{N+1}{2})(a_m+jb_m)e^{j2\pi f_m(2-\frac{N+1}{2})} \\ \vdots \\ j2\pi(N-\frac{N+1}{2})(a_m+jb_m)e^{j2\pi f_m(N-\frac{N+1}{2})} \end{bmatrix})$$

$$= \frac{8\pi^2}{\sigma^2} Re(\sum_{n=1}^{N} (a_l-jb_l)(a_m+jb_m)(n-\frac{N+1}{2})^2 e^{j2\pi(n-\frac{N+1}{2})\Delta f}) \tag{11}$$

$$
\begin{aligned}
&l = m,\\
FIM_{lm} =& \frac{8|c_l|^2\pi^2}{\sigma^2}\sum_{n=1}^{N}(n-\frac{N+1}{2})^2 = \frac{8\pi^2}{\sigma^2}\sum_{n=1}^{N}(n^2-n(N+1)+(\frac{N+1}{2})^2)\\
=&\frac{8|c_l|^2\pi^2}{\sigma^2}(\frac{N(N+1)(2N+1)}{6}-\frac{(N+1)^2N}{2}+\frac{N(N+1)^2}{4})\\
=&\frac{2|c_l|^2\pi^2N(N^2-1)}{3\sigma^2}\\
&l \neq m,\\
FIM_{lm} =&\frac{8\pi^2}{\sigma^2}Re(\sum_{n=1}^{N}(a_l-jb_l)(a_m+jb_m)(n-\frac{N+1}{2})^2e^{j2\pi(n-\frac{N+1}{2})\Delta f})\\
=&\frac{8\pi^2}{\sigma^2}Re[c_l^*c_me^{-j\pi(N+3)\Delta f}\sum_{n=0}^{N-1}(n-\frac{N+3}{2})^2e^{j2\pi n\Delta f}]\\
=&\frac{8\pi^2}{\sigma^2}Re[c_l^*c_me^{-j\pi(N+3)\Delta f}(\sum_{n=0}^{N-1}n^2e^{j2\pi n\Delta f}+(\frac{N+3}{2})\sum_{n=0}^{N-1}ne^{j2\pi n\Delta f}+(\frac{N+3}{2})^2\sum_{n=0}^{N-1}e^{j2\pi n\Delta f})]\\
=&\frac{8\pi^2}{\sigma^2}Re[c_l^*c_me^{-j\pi(N+3)\Delta f}(-f''(2\pi\Delta f)-j(\frac{N+3}{2})f'(2\pi\Delta f)+(\frac{N+3}{2})^2f(2\pi\Delta f))]\\
&\text{where} f(x) = \frac{\sin(\frac{Nx}{2})}{\sin(\frac{x}{2})}e^{j\frac{(N-1)}{2}x}
\end{aligned}
\tag{12}
$$

2.4 FIM for a_l and b_m

$$
\begin{aligned}
&2Re[\frac{\partial\mu^H}{a_l}C_y^{-1}\frac{\partial\mu}{b_m}] = 2Re(\begin{bmatrix} e^{j2\pi f_l(1-\frac{N+1}{2})}\\ e^{j2\pi f_l(2-\frac{N+1}{2})}\\ \vdots \\ e^{j2\pi f_l(N-\frac{N+1}{2})}\end{bmatrix}^H \sigma^{-2}\begin{bmatrix} je^{j2\pi f_m(1-\frac{N+1}{2})}\\ je^{j2\pi f_m(2-\frac{N+1}{2})}\\ \vdots \\ je^{j2\pi f_m(N-\frac{N+1}{2})}\end{bmatrix}) =\\
=&\frac{2}{\sigma^2}Re(j\sum_{n=1}^{N}e^{j2\pi(n-\frac{N+1}{2})\Delta f})\\
=&\frac{2}{\sigma^2}Re(je^{-j(N+3)\pi\Delta f}\sum_{n=0}^{N-1}e^{j2\pi n\Delta f})\\
&FIM_{lm} = \begin{cases} 0, & \text{if } m=1,\\ \frac{2}{\sigma^2}\frac{\sin(N\pi\Delta f)}{\sin(\pi\Delta f)}\sin(4\pi\Delta f), & \text{otherwise.}\end{cases}
\end{aligned}
\tag{13}
$$

2.5 FIM for a_l and f_m

$$2Re[\frac{\partial\mu^H}{a_l}C_y^{-1}\frac{\partial\mu}{f_m}] = 2Re(\begin{bmatrix} e^{j2\pi f_l(1-\frac{N+1}{2})} \\ e^{j2\pi f_l(2-\frac{N+1}{2})} \\ \vdots \\ e^{j2\pi f_l(N-\frac{N+1}{2})} \end{bmatrix}^H \sigma^{-2} \begin{bmatrix} j(a_m+jb_m)2\pi(1-\frac{N+1}{2})e^{j2\pi f_m(1-\frac{N+1}{2})} \\ j(a_m+jb_m)2\pi(2-\frac{N+1}{2})e^{j2\pi f_m(2-\frac{N+1}{2})} \\ \vdots \\ j(a_m+jb_m)2\pi(N-\frac{N+1}{2})e^{j2\pi f_m(N-\frac{N+1}{2})} \end{bmatrix})$$

$$=\frac{2}{\sigma^2}Re((a_m+jb_m)\sum_{n=1}^{N} j2\pi(n-\frac{N+1}{2})e^{j2\pi(n-\frac{N+1}{2})\Delta f})$$

$$=\frac{4\pi}{\sigma^2}Re[c_m\sum_{n=0}^{N-1} j(n-\frac{N+3}{2})e^{j2\pi(n-\frac{N+3}{2})\Delta f}]$$

$$=\frac{4\pi}{\sigma^2}Re[c_m e^{-j\pi(N+3)\Delta f}(\sum_{n=0}^{N-1} jne^{j2\pi n\Delta f}-\frac{N+3}{2})\sum_{n=0}^{N-1} e^{j2\pi n\Delta f})]$$

$$FIM_{lm} = \begin{cases} \frac{4\pi}{\sigma^2}\sum_{n=1}^{N} -b_m(n-\frac{N+1}{2})=0, & \text{if } m=1, \\ \frac{4\pi}{\sigma^2}Re[c_m e^{-j\pi(N+3)\Delta f}(f'(2\pi\Delta f)-\frac{N+3}{2}f(2\pi\Delta f)] & \text{otherwise, Here f(x)} = \frac{\sin(\frac{Nx}{2})}{\sin(\frac{x}{2})}e^{j\frac{(N-1)}{2}x} \end{cases} \quad (14)$$

2.6 FIM for b_l and f_m

$$2Re[\frac{\partial\mu^H}{a_l}C_y^{-1}\frac{\partial\mu}{f_m}] = 2Re(\begin{bmatrix} je^{j2\pi f_l(1-\frac{N+1}{2})} \\ je^{j2\pi f_l(2-\frac{N+1}{2})} \\ \vdots \\ je^{j2\pi f_l(N-\frac{N+1}{2})} \end{bmatrix}^H \sigma^{-2} \begin{bmatrix} j(a_m+jb_m)2\pi(1-\frac{N+1}{2})e^{j2\pi f_m(1-\frac{N+1}{2})} \\ j(a_m+jb_m)2\pi(2-\frac{N+1}{2})e^{j2\pi f_m(2-\frac{N+1}{2})} \\ \vdots \\ j(a_m+jb_m)2\pi(N-\frac{N+1}{2})e^{j2\pi f_m(N-\frac{N+1}{2})} \end{bmatrix})$$

$$=\frac{2}{\sigma^2}Re((a_m+jb_m)\sum_{n=1}^{N} 2\pi(n-\frac{N+1}{2})e^{j2\pi(n-\frac{N+1}{2})\Delta f})$$

$$=\frac{4\pi}{\sigma^2}Re[c_m\sum_{n=0}^{N-1}(n-\frac{N+3}{2})e^{j2\pi(n-\frac{N+3}{2})\Delta f}]$$

$$=\frac{4\pi}{\sigma^2}Re[-jc_m e^{-j\pi(N+3)\Delta f}(\sum_{n=0}^{N-1} jne^{j2\pi n\Delta f}-\frac{N+3}{2})\sum_{n=0}^{N-1} e^{j2\pi n\Delta f})]$$

$$FIM_{lm} = \begin{cases} \frac{4\pi}{\sigma^2}\sum_{n=1}^{N} -b_m(n-\frac{N+1}{2})=0, & \text{if } m=1, \\ \frac{4\pi}{\sigma^2}Re[-jc_m e^{-j\pi(N+3)\Delta f}(f'(2\pi\Delta f)-\frac{N+3}{2}f(2\pi\Delta f)] & \text{otherwise, Here f(x)} = \frac{\sin(\frac{Nx}{2})}{\sin(\frac{x}{2})}e^{j\frac{(N-1)}{2}x} \end{cases} \quad (15)$$

3. Maximum Likelihood Estimator

In this project, we consider a single-frequency tone ($K = 1$) signal with only unknown frequency parameter and try to estimate this parameter by two different estimators. In this section we discuss the details of maximum likelihood frequency estimator for this problem. As aforementioned, our signal is complex and contains a real and an imaginary part. For a given observation $y(n)$ at time $t_n = (n - \frac{N+1}{2})T$, where T is sample period, the PDF distribution of real and imaginary part of $y(n)$ is a Gaussian distribution.

$$
\begin{aligned}
&y(n) = x(n) + jz(n) \quad n \in \{1, 2, \cdots, N\} \\
&y(n) \sim \mathcal{CN}(ce^{j2\pi f(n-\frac{N+1}{2})}, \sigma^2) \;\; c = re^{j\delta} \quad t_n = (n - \frac{N+1}{2})T \quad (T = 1) \;\; \omega = 2\pi f \;\; \Rightarrow \\
&\mu_n = ce^{j2\pi f(n-\frac{N+1}{2})} = re^{j(\delta+\omega t_n)} = r\cos(wt_n + \delta) + jr\sin(wt_n + \delta) = \alpha_n + j\beta_n \\
&(x(n) \sim \mathcal{N}(\alpha, \frac{\sigma^2}{2}) \;\; z(n) \sim \mathcal{N}(\beta, \frac{\sigma^2}{2})
\end{aligned}
\tag{16}
$$

Therefore, the PDf of $f_Y(Y|\theta)$ can be rewritten as follow:

$$
f(Y|\theta) = (\frac{1}{\sigma^2 2\pi})^N \exp[-\frac{1}{2\sigma^2}\sum_{n=1}^{N}|y(n) - \mu_n|^2] = (\frac{1}{\sigma^2 2\pi})^N \exp[-\frac{1}{2\sigma^2}\sum_{n=1}^{N} x(n) - \alpha_n(\theta))^2 + (z(n) - \beta_n(\theta))^2]
\tag{17}
$$

The goal of maximum likelihood estimator is to maximize the probability of $f_Y(Y|\theta)$ which is equal to maximizing $\log\ f_Y(Y|\theta)$

$$
\begin{aligned}
&\log(f_Y(Y|\theta) = const - \sum_{n=1}^{N}(x(n) - \alpha_n(\theta))^2 + (z(n) - \beta_n(\theta))^2) \\
&\max\log(f_Y(Y|\theta) \Leftrightarrow \min\sum_{n=1}^{N}(x(n) - \alpha_n(\theta))^2 + (z(n) - \beta_n(\theta))^2) = \min L(\theta) \\
&L = 2\sum(x(n)^2 + \alpha_n^2 - 2\alpha_n x(n) + z(n)^2 + \beta_n^2 - 2\beta_n z(n))
\end{aligned}
\tag{18}
$$

As described in equation (18), maximizing $\log\ f_Y(Y|\theta)$ is equal to minimizing function of $L(\theta)$, where θ is f here.

$$
\begin{aligned}
\hat{f}_{ML} =& argmin\;\; L(\theta) = argmin\;\; \sum_n(\alpha_n^2(\theta) + \beta_n^2(\theta)) - 2\sum_n(\alpha_n(\theta)x(n) + \beta_n(\theta)z(n)) = \\
& argmin\;\; \sum_n(\alpha_n(\theta)^2 + \beta_n(\theta)^2) - 2\sum_n(\alpha_n(\theta)x(n) + \beta_n(\theta)z(n)) = \\
& argmin\;\; N|c|^2 - 2\sum_n(\alpha_n(\theta)x(n) + \beta_n(\theta)z(n))
\end{aligned}
\tag{19}
$$

As shown in equation (19), part of $L(\theta)$ is not function of θ. Therefore we use $L_0(\theta)$ function to simplify maximum likelihood problem.

$$
\begin{aligned}
L_0 &= = -2\sum_n(\alpha_n(\theta)x(n) + \beta_n(\theta)z(n) = -2Re[e^{-j\delta}A(f)] \\
A(f) &= \frac{1}{N}\sum_n y(n)e^{-j2\pi f t_n}
\end{aligned} \tag{20}
$$

Note that the discrete Fourier transform (DFT) of vector Y is the set of complex numbers as following:

$$
Y_F = \frac{1}{N}\sum_n y(n)e^{-\frac{j2\pi nk}{N}} = \frac{1}{N}\sum_n y(n)e^{-j2\pi t_n k} = A_k(k) \quad (k = 1, 2, \cdots, N) \tag{21}
$$

where A_k denotes k^{th} coefficient of DFT and $y(n)$ denotes n^{th} sample of time series. Therefore, function $L_0(f)$ can be formulated by Discrete Fourier Transform of observations $(A(f))$. In other words, to minimize $L_0(f)$, we have to maximize $A(f)$. Accordingly, maximum likelihood estimator for the frequency is:

$$
\hat{f}_{ML} = argmax_{\frac{2\pi kf}{N}\in[0,2\pi]} \;\; |A(\frac{2\pi kf}{N})| \tag{22}
$$

4. MoM Estimator

The other estimator that we used in this project is MoM estimator. The function that we used for MoM estimator is $\hat{R}(1) = \frac{1}{n-1}\sum_{n=1}^{N-1} y^*(n)y(n+1)$. To define a MoM estimator, we first compute the $E(\hat{R}(1))$.

Let $y(n) = ce^{j2\pi f(n-\frac{N+1}{2})} = re^{j(2\pi f(n-\frac{N+1}{2})+\delta)}$

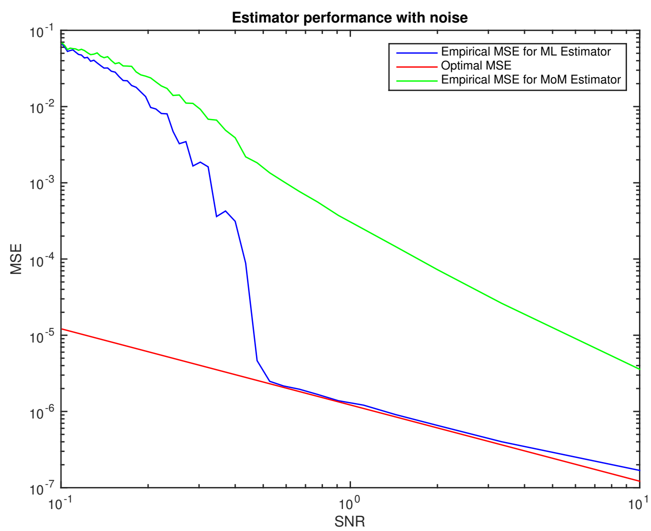
$$
\begin{aligned}
E[R(1)] &= E[\frac{1}{N-1}\sum_{n=1}^{N-1} y^*(n)y(n+1)] = \frac{1}{N-1}E[\sum_{n=1}^{N-1} y^*(n)y(n+1)] = \\
&\frac{1}{N-1}E[\sum_{n=1}^{N-1} re^{-j(2\pi f(n-\frac{N+1}{2})+\delta)} \times re^{j(2\pi f(n+1-\frac{N+1}{2})+\delta)}] = \\
&\frac{1}{N-1}E[\sum_{n=1}^{N-1} r^2 e^{j2\pi f}] = r^2E[e^{j2\pi f}] = r^2e^{j2\pi f} \Rightarrow arg\ E[\hat{R}(1)] = 2\pi f
\end{aligned} \tag{23}
$$

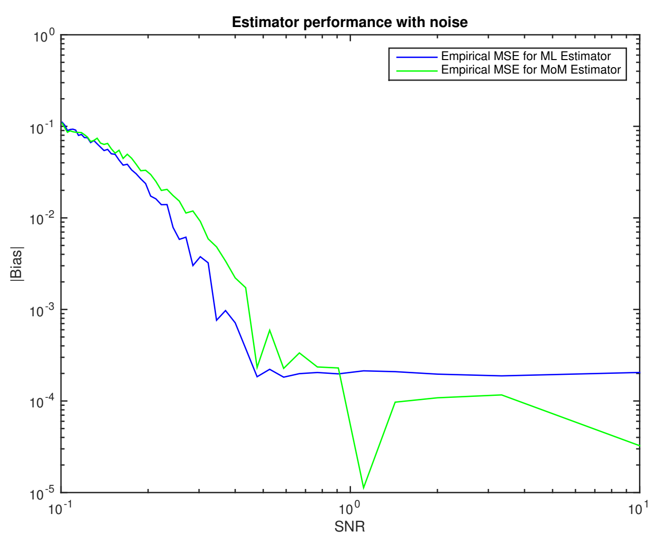
Therefore , the MoM estimator of frequency is definde by:

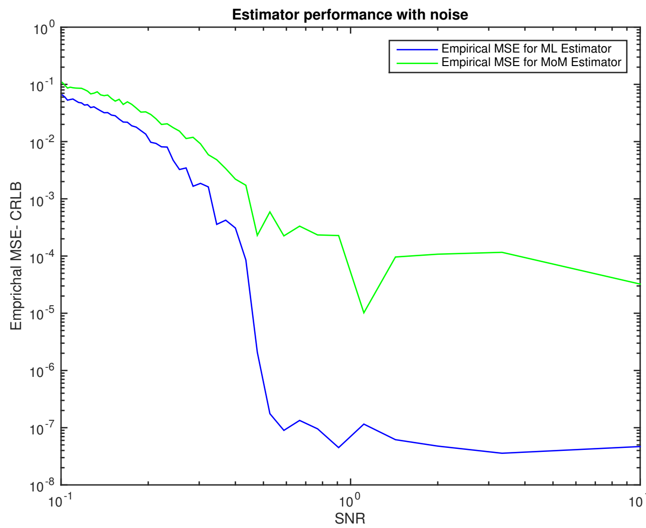
$$
\hat{f}_{MOM} = \frac{1}{2\pi}\hat{R}(1) \tag{24}
$$

5. Maximum Likelihood and MoM Estimators' Performance

To evaluate the Maximum Likelihood and MoM estimators, we considered a single-frequency tone, and took a sample set with size of 50 ($N = 50$), and set the parameter frequency as $f_1 = 0.25$ and $c_1 = 1$. We used 500 Monte-Carlo runs to compute the ML estimate of f for the realization at each run. From the 500 estimates, we computed the empirical MSE. We repeated this process while varying σ^2 from 0.1 to 10 and plot the empirical MSE of the ML estimator of f against σ^2 on a logarithmic scale (for both σ^2 and the MSE). To investigate the performance of each estimator, our plot includes CRLB graph as a function of σ^2. Also, in Figure (5) , we plot the empirical bias of the ML estimator against σ^2.







6. Conclusion

In this project we analyzed the performance of Maximum likelihood and MoM estimators with SNR of single frequency estimation with presence of complex gaussian noise.

Results shows that Maximum Likelihood estimator almost achieves the Cramer Rao Lower Bound (Liao (2011)) for a large range of SNR while MoM Estimator has a much higher MSE for even high SNR values. When SNR is below than $10^{-0.5}$ we can see that performance of MLE degrades. However by increasing the number of observations and/or by using FFT with zero padding we could increase the performance of MSE.

Analysis of bias of the two estimators shows that both estimators are unbiased.

References

Yizheng Liao. *Phase and Frequency Estimation: High-Accuracy and Low-Complexity Techniques*. PhD thesis, Worcester Polytechnic Institute, 2011.

MK Steven. *Fundamentals of Statistical Processing: Estimation Theory*. Prectice Hall, 1993.

7. Appendix

```
clc; clear all;
N=50;
f=0.25; c=1;
MC=5000;
SmoothFactor=100;

Emprical_MSE = [];
Empirical_Bias = [];
Empirical_Bias_MoM=[];
Emprical_MSE_MoM=[];

for sigma_square=0.1:0.2:10
    estimates_List=[];
    estimates_ListMoM=[];
    for i=1:MC

        % Generating N number of Measurements
        Y=zeros(N,1);
        for n=1:N
            Y(n,1)=normrnd(cos(2*pi*f*( n-(N+1)/2)),sqrt(0.5*sigma_square))+norm
        end

        %Calculating k that maximizes Aw
        omega_estimation=0;
        K_hat = 0;


        [M,K_hat]=max(fft(Y,SmoothFactor*N));


        estimates_List=[estimates_List;K_hat/(SmoothFactor*N)];

        R_hat=0;
        for(i=1:N-1)
            R_hat =R_hat + conj(Y(i))*Y(i+1);
        end
        R_hat = R_hat/(N-1);
        R_arg = angle(R_hat);
        estimates_ListMoM=[estimates_ListMoM, R_arg/(2*pi)];
    end

    % Computing empirical variance and Bias for ML estimator
```

```
    bias = abs(mean(estimates_List)-f) ;
    Empirical_Bias = [Empirical_Bias;bias];
    mse=var(estimates_List)+bias^2;
    Emprical_MSE=[Emprical_MSE;mse];


    % Computing empirical variance and Bias for ML estimator
     bias =abs( mean(estimates_ListMoM)-f);
    Empirical_Bias_MoM = [Empirical_Bias_MoM;bias];
    mse=var(estimates_ListMoM)+bias^2;
    Emprical_MSE_MoM=[Emprical_MSE_MoM;mse];
    % ML bias


end

sigma_square = 0.1:0.2:10;

sigma_square=1./sigma_square;
figure
loglog(sigma_square,Emprical_MSE','color','blue')
%plot(log(sigma_square),semilogx(Emprical_MSE'),'color','blue');
hold on
loglog(sigma_square,OPT_MSE,'color','red')
loglog(sigma_square,Emprical_MSE_MoM','color','green')
%plot(log(sigma_square),log(OPT_MSE),'color','red');
title('Estimator performance with noise')
xlabel('SNR')
ylabel('MSE')
legend('Empirical MSE for ML Estimator','Optimal MSE','Empirical MSE for MoM Est
savefig('MSESNRloglog.fig')
hold off


figure
loglog(sigma_square,(Empirical_Bias.^2)./Emprical_MSE,'color','blue')
%plot(log(sigma_square),semilogx(Emprical_MSE'),'color','blue');
hold on
%loglog(sigma_square,OPT_MSE,'color','red')
loglog(sigma_square,(Empirical_Bias_MoM.^2)./Emprical_MSE_MoM,'color','green')
%plot(log(sigma_square),log(OPT_MSE),'color','red');
title('Estimator performance with noise')
xlabel('SNR')
ylabel('Bias^2/MSE')
legend('Empirical MSE for ML Estimator', 'Empirical MSE for MoM Estimator')
```

```
savefig('BiasOverMSEloglog.fig')
hold off



figure
loglog(sigma_square,Empirical_Bias,'color','blue')
%plot(log(sigma_square),semilogx(Emprical_MSE'),'color','blue');
hold on
%loglog(sigma_square,OPT_MSE,'color','red')
loglog(sigma_square,Empirical_Bias_MoM,'color','green')
%plot(log(sigma_square),log(OPT_MSE),'color','red');
title('Estimator performance with noise')
xlabel('SNR')
ylabel('|Bias|')
legend('Empirical MSE for ML Estimator', 'Empirical MSE for MoM Estimator')
savefig('Biasloglog.fig')
hold off



figure
loglog(sigma_square,Emprical_MSE'-OPT_MSE,'color','blue')
%plot(log(sigma_square),semilogx(Emprical_MSE'),'color','blue');
hold on
%loglog(sigma_square,OPT_MSE,'color','red')
loglog(sigma_square,Empirical_Bias_MoM'-OPT_MSE,'color','green')
%plot(log(sigma_square),log(OPT_MSE),'color','red');
title('Estimator performance with noise')
xlabel('SNR')
ylabel('\sigma^2/MSE')
legend('Empirical MSE for ML Estimator', 'Empirical MSE for MoM Estimator')
savefig('Mse_Opt.fig')
hold off
```